

# PARI-GP Reference Card

(PARI-GP version 2.3.0)

Note: optional arguments are surrounded by braces {}.

## Starting & Stopping GP

to enter GP, just type its name:

`gp`

to exit GP, type

`\q` or `quit`

## Help

describe function

`?function`

extended description

`??keyword`

list of relevant help topics

`???pattern`

## Input/Output & Defaults

output previous line, the lines before

`%, %', %'', etc.`

output from line  $n$

`%n`

separate multiple statements on line

`;`

extend statement on additional lines

`\`

extend statements on several lines

`{seq1; seq2};`

comment

`/* ... */`

one-line comment, rest of line ignored

`\\ ...`

set default  $d$  to  $val$

`default({d},{val},flag)`

mimic behaviour of GP 1.39

`default(compatible,3)`

## Metacommands

toggle timer on/off

`#`

print time for last result

`##`

print  $%n$  in raw format

`\a n`

print  $%n$  in pretty format

`\b n`

print defaults

`\d`

set debug level to  $n$

`\g n`

set memory debug level to  $n$

`\gm n`

enable/disable logfile

`\l {filename}`

print  $%n$  in pretty matrix format

`\m`

set output mode (raw, default, prettyprint)

`\o n`

set  $n$  significant digits

`\p n`

set  $n$  terms in series

`\ps n`

quit GP

`\q`

print the list of PARI types

`\t`

print the list of user-defined functions

`\u`

read file into GP

`\r filename`

write  $%n$  to file

`\w n filename`

## GP Within Emacs

to enter GP from within Emacs:

`M-x gp, C-u M-x gp`

word completion

`(TAB)`

help menu window

`M-\c`

describe function

`M-?`

display  $\TeX$ 'd PARI manual

`M-x gpman`

set prompt string

`M-\p`

break line at column 100, insert `\`

`M-\\`

PARI metacommand `\letter`

`M-\letter`

## Reserved Variable Names

$\pi = 3.14159\dots$

`Pi`

Euler's constant  $= .57721\dots$

`Euler`

square root of  $-1$

`I`

big-oh notation

`O`

## PARI Types & Input Formats

`t_INT`. Integers

$\pm n$

`t_REAL`. Real Numbers

$\pm n.ddd$

`t_INTMOD`. Integers modulo  $m$

`Mod(n,m)`

`t_FRAC`. Rational Numbers

$n/m$

`t_COMPLEX`. Complex Numbers

$x + y * I$

`t_PADIC`.  $p$ -adic Numbers

$x + O(p^k)$

`t_QUAD`. Quadratic Numbers

$x + y * \text{quadgen}(D)$

`t_POLMOD`. Polynomials modulo  $g$

`Mod(f,g)`

`t_POL`. Polynomials

$a * x^n + \dots + b$

`t_SER`. Power Series

$f + O(x^k)$

`t_QFI/t_QFR`. Imag/Real bin. quad. forms

`Qfb(a,b,c,{d})`

`t_RFRAC`. Rational Functions

$f/g$

`t_VEC/t_COL`. Row/Column Vectors

$[x,y,z], [x,y,z]~$

`t_MAT`. Matrices

$[x,y;z,t;u,v]$

`t_LIST`. Lists

`List([x,y,z])`

`t_STR`. Strings

`"aaa"`

## Standard Operators

basic operations

`+, -, *, /, ^`

`i=i+1, i=i-1, i=i*j, ...`

`i++, i--, i*=j,...`

euclidean quotient, remainder

$x \backslash y, x \backslash y, x \backslash y, \text{divrem}(x,y)$

shift  $x$  left or right  $n$  bits

$x << n, x >> n$  or `shift(x,n)`

comparison operators

`<=, <, >=, >, ==, !=`

boolean operators (or, and, not)

`||, &&, !`

sign of  $x = -1, 0, 1$

`sign(x)`

maximum/minimum of  $x$  and  $y$

`max, min(x,y)`

integer or real factorial of  $x$

$x!$  or `factorial(x)`

derivative of  $f$  w.r.t.  $x$

$f'$

## Conversions

### Change Objects

to vector, matrix, set, list, string

`Col/Vec,Mat,Set,List,Str`

create PARI object ( $x \bmod y$ )

`Mod(x,y)`

make  $x$  a polynomial of  $v$

`Pol(x,{v})`

as above, starting with constant term

`Polrev(x,{v})`

make  $x$  a power series of  $v$

`Ser(x,{v})`

PARI type of object  $x$

`type(x,{t})`

object  $x$  with precision  $n$

`prec(x,{n})`

evaluate  $f$  replacing vars by their value

`eval(f)`

### Select Pieces of an Object

length of  $x$

`#x` or `length(x)`

$n$ -th component of  $x$

`component(x,n)`

$n$ -th component of vector/list  $x$

$x[n]$

$(m,n)$ -th component of matrix  $x$

$x[m,n]$

row  $m$  or column  $n$  of matrix  $x$

$x[m,], x[,n]$

numerator of  $x$

`numerator(x)`

lowest denominator of  $x$

`denominator(x)`

### Conjugates and Lifts

conjugate of a number  $x$

`conj(x)`

conjugate vector of algebraic number  $x$

`conjvec(x)`

norm of  $x$ , product with conjugate

`norm(x)`

square of  $L^2$  norm of vector  $x$

`norml2(x)`

lift of  $x$  from Mods

`lift, centerlift(x)`

## Random Numbers

random integer between 0 and  $N - 1$

`random({N})`

get random seed

`getrand()`

set random seed to  $s$

`setrand(s)`

## Lists, Sets & Sorting

sort  $x$  by  $k$ th component

`vecsort(x,{k},{fl=0})`

**Sets** (= row vector of strings with strictly increasing entries)

intersection of sets  $x$  and  $y$  `setintersect(x,y)`

set of elements in  $x$  not belonging to  $y$  `setminus(x,y)`

union of sets  $x$  and  $y$  `setunion(x,y)`

look if  $y$  belongs to the set  $x$  `setsearch(x,y,flag)`

### Lists

create empty list of maximal length  $n$  `listcreate(n)`

delete all components of list  $l$  `listkill(l)`

append  $x$  to list  $l$  `listput(l,x,{i})`

insert  $x$  in list  $l$  at position  $i$  `listinsert(l,x,i)`

sort the list  $l$  `listsort(l,flag)`

## Programming & User Functions

**Control Statements** ( $X$ : formal parameter in expression  $seq$ )

eval.  $seq$  for  $a \leq X \leq b$  `for(X=a,b,seq)`

eval.  $seq$  for  $X$  dividing  $n$  `fordiv(n,X,seq)`

eval.  $seq$  for primes  $a \leq X \leq b$  `forprime(X=a,b,seq)`

eval.  $seq$  for  $a \leq X \leq b$  stepping  $s$  `forstep(X=a,b,s,seq)`

multivariable for `forvec(X=v,seq)`

if  $a \neq 0$ , evaluate  $seq_1$ , else  $seq_2$  `if(a,{seq1},{seq2})`

evaluate  $seq$  until  $a \neq 0$  `until(a,seq)`

while  $a \neq 0$ , evaluate  $seq$  `while(a,seq)`

exit  $n$  innermost enclosing loops `break({n})`

start new iteration of  $n$ th enclosing loop `next({n})`

return  $x$  from current subroutine `return(x)`

error recovery (try  $seq_1$ ) `trap({err},{seq2},{seq1})`

### Input/Output

prettyprint args with/without newline `printp(), printp1()`

print args with/without newline `print(), print1()`

read a string from keyboard `input()`

reorder priority of variables  $x,y,z$  `reorder({[x,y,z]})`

output  $args$  in  $\TeX$  format `printtex(args)`

write  $args$  to file `write, write1, writetex(file,args)`

read file into GP `read({file})`

### Interface with User and System

allocates a new stack of  $s$  bytes `allocatemem({s})`

execute system command  $a$  `system(a)`

as above, feed result to GP `extern(a)`

install function from library `install(f,code,{gpf},{lib})`

alias  $old$  to  $new$  `alias(new,old)`

new name of function  $f$  in GP 2.0 `whatnow(f)`

### User Defined Functions

`name(formal vars) = local(local vars); seq`

`struct.member = seq`

kill value of variable or function  $x$  `kill(x)`

declare global variables `global(x,...)`

## Iterations, Sums & Products

numerical integration `intnum(X=a,b,expr,flag)`

sum  $expr$  over divisors of  $n$  `sumdiv(n,X,expr)`

sum  $X = a$  to  $X = b$ , initialized at  $x$  `sum(X=a,b,expr,{x})`

sum of series  $expr$  `suminf(X=a,expr)`

sum of alternating/positive series `sumalt, sumpos`

product  $a \leq X \leq b$ , initialized at  $x$  `prod(X=a,b,expr,{x})`

product over primes  $a \leq X \leq b$  `prodeuler(X=a,b,expr)`

infinite product  $a \leq X \leq \infty$  `prodinf(X=a,expr)`

real root of  $expr$  between  $a$  and  $b$  `solve(X=a,b,expr)`

Vectors & Matrices

dimensions of matrix $x$	<code>matsize(<math>x</math>)</code>
concatenation of $x$ and $y$	<code>concat(<math>x, \{y\}</math>)</code>
extract components of $x$	<code>vecextract(<math>x, y, \{z\}</math>)</code>
transpose of vector or matrix $x$	<code>mattranspose(<math>x</math>)</code> or <code>x-</code>
adjoint of the matrix $x$	<code>matadjoin(<math>x</math>)</code>
eigenvectors of matrix $x$	<code>mateigen(<math>x</math>)</code>
characteristic polynomial of $x$	<code>charpoly(<math>x, \{v\}, flag</math>)</code>
minimal polynomial of $x$	<code>minpoly(<math>x, \{v\}</math>)</code>
trace of matrix $x$	<code>trace(<math>x</math>)</code>

Constructors & Special Matrices

row vec. of $expr$ eval'd at $1 \leq i \leq n$	<code>vector(<math>n, \{i\}, \{expr\}</math>)</code>
col. vec. of $expr$ eval'd at $1 \leq i \leq n$	<code>vectorv(<math>n, \{i\}, \{expr\}</math>)</code>
matrix $1 \leq i \leq m, 1 \leq j \leq n$	<code>matrix(<math>m, n, \{i\}, \{j\}, \{expr\}</math>)</code>
diagonal matrix whose diag. is $x$	<code>matdiagonal(<math>x</math>)</code>
$n \times n$ identity matrix	<code>matid(<math>n</math>)</code>
Hessenberg form of square matrix $x$	<code>mathess(<math>x</math>)</code>
$n \times n$ Hilbert matrix $H_{ij} = (i + j - 1)^{-1}$	<code>mathilbert(<math>n</math>)</code>
$n \times n$ Pascal triangle $P_{ij} = \binom{i}{j}$	<code>matpascal(<math>n - 1</math>)</code>
companion matrix to polynomial $x$	<code>matcompanion(<math>x</math>)</code>

Gaussian elimination

determinant of matrix $x$	<code>matdet(<math>x, flag</math>)</code>
kernel of matrix $x$	<code>matker(<math>x, flag</math>)</code>
intersection of column spaces of $x$ and $y$	<code>matintersect(<math>x, y</math>)</code>
solve $M * X = B$ ( $M$ invertible)	<code>matsolve(<math>M, B</math>)</code>
as solve, modulo $D$ (col. vector)	<code>matsolvemod(<math>M, D, B</math>)</code>
one sol of $M * X = B$	<code>matinverseimage(<math>M, B</math>)</code>
basis for image of matrix $x$	<code>matimage(<math>x</math>)</code>
supplement columns of $x$ to get basis	<code>mat supplement(<math>x</math>)</code>
rows, cols to extract invertible matrix	<code>matindexrank(<math>x</math>)</code>
rank of the matrix $x$	<code>matrank(<math>x</math>)</code>

Lattices & Quadratic Forms

upper triangular Hermite Normal Form	<code>mathnf(<math>x</math>)</code>
HNF of $x$ where $d$ is a multiple of $\det(x)$	<code>mathnfmod(<math>x, d</math>)</code>
elementary divisors of $x$	<code>matsnf(<math>x</math>)</code>
LLL-algorithm applied to columns of $x$	<code>qflll(<math>x, flag</math>)</code>
like <code>qflll</code> , $x$ is Gram matrix of lattice	<code>qflllgram(<math>x, flag</math>)</code>
LLL-reduced basis for kernel of $x$	<code>matkerint(<math>x</math>)</code>
<b>Z</b> -lattice $\longleftrightarrow$ <b>Q</b> -vector space	<code>matrixqz(<math>x, p</math>)</code>
signature of quad form ${}^t y * x * y$	<code>qfsign(<math>x</math>)</code>
decomp into squares of ${}^t y * x * y$	<code>qfgaussred(<math>x</math>)</code>
find up to $m$ sols of ${}^t y * x * y \leq b$	<code>qfminim(<math>x, b, m</math>)</code>
$v, v[i] :=$ number of sols of ${}^t y * x * y = i$	<code>qfrep(<math>x, B, flag</math>)</code>
eigenvals/eigenvecs for real symmetric $x$	<code>qfjacobi(<math>x</math>)</code>

Formal & p-adic Series

truncate power series or $p$ -adic number	<code>truncate(<math>x</math>)</code>
valuation of $x$ at $p$	<code>valuation(<math>x, p</math>)</code>
<b>Dirichlet and Power Series</b>	
Taylor expansion around 0 of $f$ w.r.t. $x$	<code>taylor(<math>f, x</math>)</code>
$\sum a_k b_k t^k$ from $\sum a_k t^k$ and $\sum b_k t^k$	<code>serconvol(<math>x, y</math>)</code>
$f = \sum a_k * t^k$ from $\sum (a_k / k!) * t^k$	<code>serlaplace(<math>f</math>)</code>
reverse power series $F$ so $F(f(x)) = x$	<code>serreverse(<math>f</math>)</code>
Dirichlet series multiplication / division	<code>dirmul, dirdiv(<math>x, y</math>)</code>
Dirichlet Euler product ( $b$ terms)	<code>direuler(<math>p = a, b, expr</math>)</code>

p-adic Functions

Teichmuller character of $x$	<code>teichmuller(<math>x</math>)</code>
Newton polygon of $f$ for prime $p$	<code>newtonpoly(<math>f, p</math>)</code>

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Polynomials & Rational Functions

degree of $f$	<code>poldegree(<math>f</math>)</code>
coefficient of degree $n$ of $f$	<code>polcoeff(<math>f, n</math>)</code>
round coeffs of $f$ to nearest integer	<code>round(<math>f, \{&amp;e\}</math>)</code>
gcd of coefficients of $f$	<code>content(<math>f</math>)</code>
replace $x$ by $y$ in $f$	<code>subst(<math>f, x, y</math>)</code>
discriminant of polynomial $f$	<code>poldisc(<math>f</math>)</code>
resultant of $f$ and $g$	<code>polresultant(<math>f, g, flag</math>)</code>
as above, give $[u, v, d]$ , $xu + yv = d$	<code>bezoutres(<math>x, y</math>)</code>
derivative of $f$ w.r.t. $x$	<code>deriv(<math>f, x</math>)</code>
formal integral of $f$ w.r.t. $x$	<code>intformal(<math>f, x</math>)</code>
reciprocal poly $x^{\deg f} f(1/x)$	<code>polrecip(<math>f</math>)</code>
interpol. pol. eval. at $a$	<code>polinterpolate(<math>X, \{Y\}, \{a\}, \{&amp;e\}</math>)</code>
initialize $t$ for Thue equation solver	<code>thueinit(<math>f</math>)</code>
solve Thue equation $f(x, y) = a$	<code>thue(<math>t, a, \{sol\}</math>)</code>

Roots and Factorization

number of real roots of $f$ , $a < x \leq b$	<code>polsturm(<math>f, \{a\}, \{b\}</math>)</code>
complex roots of $f$	<code>polroots(<math>f</math>)</code>
symmetric powers of roots of $f$ up to $n$	<code>polsym(<math>f, n</math>)</code>
roots of $f$ mod $p$	<code>polrootsmod(<math>f, p, flag</math>)</code>
factor $f$	<code>factor(<math>f, \{lim\}</math>)</code>
factorization of $f$ mod $p$	<code>factormod(<math>f, p, flag</math>)</code>
factorization of $f$ over $\mathbb{F}_{p^a}$	<code>factorff(<math>f, p, a</math>)</code>
$p$ -adic fact. of $f$ to prec. $r$	<code>factorpadic(<math>f, p, r, flag</math>)</code>
$p$ -adic roots of $f$ to prec. $r$	<code>polrootspadic(<math>f, p, r</math>)</code>
$p$ -adic root of $f$ cong. to $a$ mod $p$	<code>padicappr(<math>f, a</math>)</code>
Newton polygon of $f$ for prime $p$	<code>newtonpoly(<math>f, p</math>)</code>

Special Polynomials

$n$ th cyclotomic polynomial in var. $v$	<code>polcyclo(<math>n, \{v\}</math>)</code>
$d$ -th degree subfield of $\mathbb{Q}(\zeta_n)$	<code>polsubcyclo(<math>n, d, \{v\}</math>)</code>
$n$ -th Legendre polynomial	<code>pollegendre(<math>n</math>)</code>
$n$ -th Tchebicheff polynomial	<code>pol tchebi(<math>n</math>)</code>
Zagier's polynomial of index $n, m$	<code>polzagier(<math>n, m</math>)</code>

Transcendental Functions

real, imaginary part of $x$	<code>real(<math>x</math>), imag(<math>x</math>)</code>
absolute value, argument of $x$	<code>abs(<math>x</math>), arg(<math>x</math>)</code>
square/ $n$ th root of $x$	<code>sqrtn(<math>x, n, &amp;z</math>)</code>
trig functions	<code>sin, cos, tan, cotan</code>
inverse trig functions	<code>asin, acos, atan</code>
hyperbolic functions	<code>sinh, cosh, tanh</code>
inverse hyperbolic functions	<code>asinh, acosh, atanh</code>
exponential of $x$	<code>exp(<math>x</math>)</code>
natural log of $x$	<code>ln(<math>x</math>)</code> or <code>log(<math>x</math>)</code>
gamma function $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$	<code>gamma(<math>x</math>)</code>
logarithm of gamma function	<code>lngamma(<math>x</math>)</code>
$\psi(x) = \Gamma'(x)/\Gamma(x)$	<code>psi(<math>x</math>)</code>
incomplete gamma function ( $y = \Gamma(s)$ )	<code>incgam(<math>s, x, \{y\}</math>)</code>
exponential integral $\int_x^\infty e^{-t}/t dt$	<code>eint1(<math>x</math>)</code>
error function $2/\sqrt{\pi} \int_x^\infty e^{-t^2} dt$	<code>erfc(<math>x</math>)</code>
dilogarithm of $x$	<code>dilog(<math>x</math>)</code>
$m$ th polylogarithm of $x$	<code>polylog(<math>m, x, flag</math>)</code>
$U$ -confluent hypergeometric function	<code>hyperu(<math>a, b, u</math>)</code>
$J$ -Bessel function $J_{n+1/2}(x)$	<code>besseljh(<math>n, x</math>)</code>
$K$ -Bessel function of index $nu$	<code>besselk(<math>nu, x</math>)</code>

Elementary Arithmetic Functions

vector of binary digits of $ x $	<code>binary(<math>x</math>)</code>
give bit number $n$ of integer $x$	<code>bittest(<math>x, n</math>)</code>
ceiling of $x$	<code>ceil(<math>x</math>)</code>
floor of $x$	<code>floor(<math>x</math>)</code>
fractional part of $x$	<code>frac(<math>x</math>)</code>
round $x$ to nearest integer	<code>round(<math>x, \{&amp;e\}</math>)</code>
truncate $x$	<code>truncate(<math>x, \{&amp;e\}</math>)</code>
gcd/LCM of $x$ and $y$	<code>gcd(<math>x, y</math>), lcm(<math>x, y</math>)</code>
gcd of entries of a vector/matrix	<code>content(<math>x</math>)</code>
<b>Primes and Factorization</b>	
add primes in $v$ to the prime table	<code>addprimes(<math>v</math>)</code>
the $n$ th prime	<code>prime(<math>n</math>)</code>
vector of first $n$ primes	<code>primes(<math>n</math>)</code>
smallest prime $\geq x$	<code>nextprime(<math>x</math>)</code>
largest prime $\leq x$	<code>precprime(<math>x</math>)</code>
factorization of $x$	<code>factor(<math>x, \{lim\}</math>)</code>
reconstruct $x$ from its factorization	<code>factorback(<math>fa, \{nf\}</math>)</code>

Divisors

number of distinct prime divisors	<code>omega(<math>x</math>)</code>
number of prime divisors with mult	<code>bigomega(<math>x</math>)</code>
number of divisors of $x$	<code>numdiv(<math>x</math>)</code>
row vector of divisors of $x$	<code>divisors(<math>x</math>)</code>
sum of ( $k$ -th powers of) divisors of $x$	<code>sigma(<math>x, \{k\}</math>)</code>

Special Functions and Numbers

binomial coefficient $\binom{x}{y}$	<code>binomial(<math>x, y</math>)</code>
Bernoulli number $B_n$ as real	<code>bernreal(<math>n</math>)</code>
Bernoulli vector $B_0, B_2, \dots, B_{2n}$	<code>bernvec(<math>n</math>)</code>
$n$ th Fibonacci number	<code>fibonacci(<math>n</math>)</code>
number of partitions of $n$	<code>numbpart(<math>n</math>)</code>
Euler $\phi$ -function	<code>eulerphi(<math>x</math>)</code>
Möbius $\mu$ -function	<code>moebius(<math>x</math>)</code>
Hilbert symbol of $x$ and $y$ (at $p$ )	<code>hilbert(<math>x, y, \{p\}</math>)</code>
Kronecker-Legendre symbol $(\frac{x}{y})$	<code>kronecker(<math>x, y</math>)</code>

Miscellaneous

integer or real factorial of $x$	<code>x!</code> or <code>fact(<math>x</math>)</code>
integer square root of $x$	<code>sqrntint(<math>x</math>)</code>
solve $z \equiv x$ and $z \equiv y$	<code>chinese(<math>x, y</math>)</code>
minimal $u, v$ so $xu + yv = \gcd(x, y)$	<code>bezout(<math>x, y</math>)</code>
multiplicative order of $x$ (intmod) (i=0)	<code>znorder(<math>x, \{o\}</math>)</code>
primitive root mod prime power $q$	<code>znprimroot(<math>q</math>)</code>
structure of $(\mathbb{Z}/n\mathbb{Z})^*$	<code>znstar(<math>n</math>)</code>
continued fraction of $x$	<code>contfrac(<math>x, \{b\}, \{lmax\}</math>)</code>
last convergent of continued fraction $x$	<code>contfracpnqn(<math>x</math>)</code>
best rational approximation to $x$	<code>bestappr(<math>x, k</math>)</code>

True-False Tests

is $x$ the disc. of a quadratic field?	<code>isfundamental(<math>x</math>)</code>
is $x$ a prime?	<code>isprime(<math>x</math>)</code>
is $x$ a strong pseudo-prime?	<code>ispseudoprime(<math>x</math>)</code>
is $x$ square-free?	<code>issquarefree(<math>x</math>)</code>
is $x$ a square?	<code>Zissquare(<math>x, \{&amp;n\}</math>)</code>
is $pol$ irreducible?	<code>polisirreducible(<math>pol</math>)</code>

Based on an earlier version by Joseph H. Silverman

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# PARI-GP Reference Card (2)

(PARI-GP version 2.3.0)

## Elliptic Curves

Elliptic curve initially given by 5-tuple  $E = [a_1, a_2, a_3, a_4, a_6]$ . Points are  $[x, y]$ , the origin is  $[0]$ .

Initialize elliptic struct.  $ell$ , i.e create `ellinit( $E, flag$ )`

$a_1, a_2, a_3, a_4, a_6, b_2, b_4, b_6, b_8, c_4, c_6, disc, j$ . This data can be recovered by typing  $ell.a1, \dots, ell.j$ . If  $fl$  omitted, also

$E$ defined over <b>R</b>	
$x$ -coords. of points of order 2	<code>ell.roots</code>
real and complex periods	<code>ell.omega</code>
associated quasi-periods	<code>ell.eta</code>
volume of complex lattice	<code>ell.area</code>
$E$ defined over $\mathbf{Q}_p,  j _p > 1$	
$x$ -coord. of unit 2 torsion point	<code>ell.roots</code>
Tate's $[u^2, u, q]$	<code>ell.tate</code>
Mestre's $w$	<code>ell.w</code>
change curve $E$ using $v = [u, r, s, t]$	<code>ellchangecurve(<math>ell, v</math>)</code>
change point $z$ using $v = [u, r, s, t]$	<code>ellchangepoint(<math>z, v</math>)</code>
cond, min mod, Tamagawa num $[N, v, c]$	<code>ellglobalred(<math>ell</math>)</code>
Kodaira type of $p$ fiber of $E$	<code>elllocalred(<math>ell, p</math>)</code>
add points $z_1 + z_2$	<code>elladd(<math>ell, z_1, z_2</math>)</code>
subtract points $z_1 - z_2$	<code>ellsub(<math>ell, z_1, z_2</math>)</code>
compute $n \cdot z$	<code>ellpow(<math>ell, z, n</math>)</code>
check if $z$ is on $E$	<code>ellisoncurve(<math>ell, z</math>)</code>
order of torsion point $z$	<code>ellorder(<math>ell, z</math>)</code>
torsion subgroup with generators	<code>elltors(<math>ell</math>)</code>
$y$ -coordinates of point(s) for $x$	<code>ellordinate(<math>ell, x</math>)</code>
canonical bilinear form taken at $z_1, z_2$	<code>ellbil(<math>ell, z_1, z_2</math>)</code>
canonical height of $z$	<code>ellheight(<math>ell, z, flag</math>)</code>
height regulator matrix for pts in $x$	<code>ellheightmatrix(<math>ell, x</math>)</code>
$p$ th coeff $a_p$ of $L$ -function, $p$ prime	<code>ellap(<math>ell, p</math>)</code>
$k$ th coeff $a_k$ of $L$ -function	<code>ellak(<math>ell, k</math>)</code>
vector of first $n$ $a_k$ 's in $L$ -function	<code>ellan(<math>ell, n</math>)</code>
$L(E, s)$ , set $A \approx 1$	<code>elllseries(<math>ell, s, \{A\}</math>)</code>
root number for $L(E, \cdot)$ at $p$	<code>ellrootno(<math>ell, \{p\}</math>)</code>
modular parametrization of $E$	<code>elltaniyama(<math>ell</math>)</code>
point $[\wp(z), \wp'(z)]$ corresp. to $z$	<code>ellztopoint(<math>ell, z</math>)</code>
complex $z$ such that $p = [\wp(z), \wp'(z)]$	<code>ellpointtoz(<math>ell, p</math>)</code>

## Elliptic & Modular Functions

arithmetic-geometric mean	<code>agm(<math>x, y</math>)</code>
elliptic $j$ -function $1/q + 744 + \dots$	<code>ellj(<math>x</math>)</code>
Weierstrass $\sigma$ function	<code>ellsigma(<math>ell, z, flag</math>)</code>
Weierstrass $\wp$ function	<code>ellwp(<math>ell, \{z\}, flag</math>)</code>
Weierstrass $\zeta$ function	<code>ellzeta(<math>ell, z</math>)</code>
modified Dedekind $\eta$ func. $\prod(1 - q^n)$	<code>eta(<math>x, flag</math>)</code>
Jacobi sine theta function	<code>theta(<math>q, z</math>)</code>
k-th derivative at $z=0$ of $\theta(q, z)$	<code>thetanullk(<math>q, k</math>)</code>
Weber's $f$ functions	<code>weber(<math>x, flag</math>)</code>
Riemann's zeta $\zeta(s) = \sum n^{-s}$	<code>zeta(<math>s</math>)</code>

## Graphic Functions

crude graph of  $expr$  between  $a$  and  $b$  `plot( $X = a, b, expr$ )`  
**High-resolution plot** (immediate plot)  
plot  $expr$  between  $a$  and  $b$  `plotth( $X = a, b, expr, flag, \{n\}$ )`  
plot points given by lists  $lx, ly$  `plotthraw( $lx, ly, flag$ )`  
terminal dimensions `plotsizes()`

### Rectwindow functions

init window  $w$ , with size  $x, y$  `plotinit( $w, x, y$ )`  
erase window  $w$  `plotkill( $w$ )`  
copy  $w$  to  $w_2$  with offset  $(dx, dy)$  `plotcopy( $w, w_2, dx, dy$ )`  
scale coordinates in  $w$  `plotscale( $w, x_1, x_2, y_1, y_2$ )`  
`plotth` in  $w$  `plotrecth( $w, X = a, b, expr, flag, \{n\}$ )`  
`plotthraw` in  $w$  `plotrecthraw( $w, data, flag$ )`  
draw window  $w_1$  at  $(x_1, y_1), \dots$  `plotdraw([[ $w_1, x_1, y_1$ ], \dots])`

### Low-level Rectwindow Functions

set current drawing color in  $w$  to  $c$  `plotcolor( $w, c$ )`  
current position of cursor in  $w$  `plotcursor( $w$ )`  
write  $s$  at cursor's position `plotstring( $w, s$ )`  
move cursor to  $(x, y)$  `plotmove( $w, x, y$ )`  
move cursor to  $(x + dx, y + dy)$  `plotrmove( $w, dx, dy$ )`  
draw a box to  $(x_2, y_2)$  `plotbox( $w, x_2, y_2$ )`  
draw a box to  $(x + dx, y + dy)$  `plotrbox( $w, dx, dy$ )`  
draw polygon `plotlines( $w, lx, ly, flag$ )`  
draw points `plotpoints( $w, lx, ly$ )`  
draw line to  $(x + dx, y + dy)$  `plotrline( $w, dx, dy$ )`  
draw point  $(x + dx, y + dy)$  `plotrpoint( $w, dx, dy$ )`

### Postscript Functions

as `plotth` `psplotth( $X = a, b, expr, flag, \{n\}$ )`  
as `plotthraw` `psplotthraw( $lx, ly, flag$ )`  
as `plotdraw` `psdraw([[ $w_1, x_1, y_1$ ], \dots])`

## Binary Quadratic Forms

create  $ax^2 + bxy + cy^2$  (distance  $d$ ) `qfb( $a, b, c, \{d\}$ )`  
reduce  $x$  ( $s = \sqrt{D}$ ,  $l = \lfloor s \rfloor$ ) `qfbred( $x, flag, \{D\}, \{l\}, \{s\}$ )`  
composition of forms  $x*y$  or `qfbnucomp( $x, y, l$ )`  
 $n$ -th power of form  $x^n$  or `qfbnupow( $x, n$ )`  
composition without reduction `qfbcompraw( $x, y$ )`  
 $n$ -th power without reduction `qfbpowraw( $x, n$ )`  
prime form of disc.  $x$  above prime  $p$  `qfbprimeform( $x, p$ )`  
class number of disc.  $x$  `qfbclassno( $x$ )`  
Hurwitz class number of disc.  $x$  `qfbhclassno( $x$ )`

## Quadratic Fields

quadratic number  $\omega = \sqrt{x}$  or  $(1 + \sqrt{x})/2$  `quadgen( $x$ )`  
minimal polynomial of  $\omega$  `quadpoly( $x$ )`  
discriminant of  $\mathbf{Q}(\sqrt{D})$  `quaddisc( $x$ )`  
regulator of real quadratic field `quadregulator( $x$ )`  
fundamental unit in real  $\mathbf{Q}(x)$  `quadunit( $x$ )`  
class group of  $\mathbf{Q}(\sqrt{D})$  `quadclassunit( $D, flag, \{t\}$ )`  
Hilbert class field of  $\mathbf{Q}(\sqrt{D})$  `quadhilbert( $D, flag$ )`  
ray class field modulo  $f$  of  $\mathbf{Q}(\sqrt{D})$  `quadrday( $D, f, flag$ )`

## General Number Fields: Initializations

A number field  $K$  is given by a monic irreducible  $f \in \mathbf{Z}[X]$ .

init number field structure  $nf$  `nfinit( $f, flag$ )`

### nf members:

polynomial defining $nf$ , $f(\theta) = 0$	<code>nf.pol</code>
number of real/complex places	<code>nf.r1, nf.r2</code>
discriminant of $nf$	<code>nf.disc</code>
$T_2$ matrix	<code>nf.t2</code>
vector of roots of $f$	<code>nf.roots</code>
integral basis of $\mathbf{Z}_K$ as powers of $\theta$	<code>nf.zk</code>
different	<code>nf.diff</code>
codifferent	<code>nf.codiff</code>
recompute $nf$ using current precision	<code>nfnewprec(<math>nf</math>)</code>
init relative $rmf$ given by $g = 0$ over $K$	<code>rnfininit(<math>nf, g</math>)</code>
init $bnf$ structure	<code>bnfininit(<math>f, flag</math>)</code>

**bnf members:** same as  $nf$ , plus

underlying $nf$	<code>bnf.nf</code>
classgroup	<code>bnf.clgp</code>
regulator	<code>bnf.reg</code>
fundamental units	<code>bnf.fu</code>
torsion units	<code>bnf.tu</code>
$[tu, fu]$	<code>bnf.tufu</code>
compute a $bnf$ from small $bnf$	<code>bnfmake(<math>sbnf</math>)</code>
add $S$ -class group and units, yield $bnf$ s	<code>bnfsunit(<math>nf, S</math>)</code>
init class field structure $bnr$	<code>bnrinit(<math>bnf, m, flag</math>)</code>

**bnr members:** same as  $bnf$ , plus

underlying $bnf$	<code>bnr.bnf</code>
structure of $(\mathbf{Z}_K/m)^*$	<code>bnr.zkst</code>

## Simple Arithmetic Invariants (nf)

Elements are rational numbers, polynomials, polmods, or column vectors (on integral basis  $nf.zk$ ).

integral basis of field def. by  $f = 0$       **nfbasis**( $f$ )  
field discriminant of field  $f = 0$       **nfdisc**( $f$ )  
reverse polmod  $a = A(X) \bmod T(X)$       **modreverse**( $a$ )  
Galois group of field  $f = 0$ ,  $\deg f \leq 11$       **polgalois**( $f$ )  
smallest poly defining  $f = 0$       **polredabs**( $f, flag$ )  
small polys defining subfields of  $f = 0$       **polred**( $f, flag, \{p\}$ )  
small polys defining suborders of  $f = 0$       **polredord**( $f$ )  
poly of degree  $\leq k$  with root  $x \in \mathbf{C}$       **algdep**( $x, k$ )  
small linear rel. on coords of vector  $x$       **lindep**( $x$ )  
are fields  $f = 0$  and  $g = 0$  isomorphic?      **nfisism**( $f, g$ )  
is field  $f = 0$  a subfield of  $g = 0$ ?      **nfisincl**( $f, g$ )  
compositum of  $f = 0$ ,  $g = 0$       **polcompositum**( $f, g, flag$ )  
basic element operations (prefix **nfelt**):

(**nfelt**)**mul**, **pow**, **div**, **diveuc**, **mod**, **divrem**, **val**  
express  $x$  on integer basis      **nfalgtobasis**( $nf, x$ )  
express element  $x$  as a polmod      **nfbasistoalg**( $nf, x$ )  
quadratic Hilbert symbol (at  $p$ )      **nfhilbert**( $nf, a, b, \{p\}$ )  
roots of  $g$  belonging to  $nf$       **nfroots**( $\{nf\}, g$ )  
factor  $g$  in  $nf$       **nfactor**( $nf, g$ )  
factor  $g$  mod prime  $pr$  in  $nf$       **nfactormod**( $nf, g, pr$ )  
number of roots of unity in  $nf$       **nfrootsof1**( $nf$ )  
conjugates of a root  $\theta$  of  $nf$       **nfgaloisconj**( $nf, flag$ )  
apply Galois automorphism  $s$  to  $x$       **nfgaloisapply**( $nf, s, x$ )  
subfields (of degree  $d$ ) of  $nf$       **nfsubfields**( $nf, \{d\}$ )

### Dedekind Zeta Function $\zeta_K$

$\zeta_K$  as Dirichlet series,  $N(I) < b$       **dirzetak**( $nf, b$ )  
init  $nfz$  for field  $f = 0$       **zetakinit**( $f$ )  
compute  $\zeta_K(s)$       **zetak**( $nfz, s, flag$ )  
Artin root number of  $K$       **bnrrootnumber**( $bnr, chi, flag$ )

## Class Groups & Units (bnf, bnr)

$a_1, \{a_2\}, \{a_3\}$  usually  $bnr, subgp$  or  $bnf, module, \{subgp\}$   
remove GRH assumption from  $bnf$       **bnfcertify**( $bnf$ )  
expo. of ideal  $x$  on class gp      **bnfisprincipal**( $bnf, x, flag$ )  
expo. of ideal  $x$  on ray class gp      **bnrisprincipal**( $bnr, x, flag$ )  
expo. of  $x$  on fund. units      **bnfisunit**( $bnf, x$ )  
as above for  $S$ -units      **bnfissunit**( $bnfs, x$ )  
fundamental units of  $bnf$       **bnfunit**( $bnf$ )  
signs of real embeddings of  $bnf.fu$       **bnfsignunit**( $bnf$ )

### Class Field Theory

ray class group structure for mod.  $m$       **bnrclass**( $bnf, m, flag$ )  
ray class number for mod.  $m$       **bnrclassno**( $bnf, m$ )  
discriminant of class field ext      **bnrdisc**( $a_1, \{a_2\}, \{a_3\}$ )  
ray class numbers,  $l$  list of mods      **bnrclassnolist**( $bnf, l$ )  
discriminants of class fields      **bnrdisclist**( $bnf, l, \{arch\}, flag$ )  
decode output from **bnrdisclist**      **bnfdecodemodule**( $nf, fa$ )  
is modulus the conductor?      **bnrisconductor**( $a_1, \{a_2\}, \{a_3\}$ )  
conductor of character  $chi$       **bnrconductorofchar**( $bnr, chi$ )  
conductor of extension      **bnrconductor**( $a_1, \{a_2\}, \{a_3\}, flag$ )  
conductor of extension def. by  $g$       **rnfconductor**( $bnf, g$ )  
Artin group of ext. def'd by  $g$       **rnfnormgroup**( $bnr, g$ )  
subgroups of  $bnr$ , index  $\leq b$       **subgrouplist**( $bnr, b, flag$ )  
rel. eq. for class field def'd by  $sub$       **rnfkummer**( $bnr, sub, \{d\}$ )  
same, using Stark units (real field)      **bnrstark**( $bnr, sub, flag$ )

## PARI-GP Reference Card (2)

(PARI-GP version 2.3.0)

### Ideals

Ideals are elements, primes, or matrix of generators in HNF.  
is  $id$  an ideal in  $nf$ ?      **nfisideal**( $nf, id$ )  
is  $x$  principal in  $bnf$ ?      **bnfisprincipal**( $bnf, x$ )  
principal ideal generated by  $x$       **idealprincipal**( $nf, x$ )  
principal idele generated by  $x$       **ideleprincipal**( $nf, x$ )  
give  $[a, b]$ , s.t.  $a\mathbf{Z}_K + b\mathbf{Z}_K = x$       **idealtwoelt**( $nf, x, \{a\}$ )  
put ideal  $a$  ( $a\mathbf{Z}_K + b\mathbf{Z}_K$ ) in HNF form      **idealhnf**( $nf, a, \{b\}$ )  
norm of ideal  $x$       **idealnrm**( $nf, x$ )  
minimum of ideal  $x$  (direction  $v$ )      **idealmin**( $nf, x, v$ )  
LLL-reduce the ideal  $x$  (direction  $v$ )      **idealred**( $nf, x, \{v\}$ )

### Ideal Operations

add ideals  $x$  and  $y$       **idealadd**( $nf, x, y$ )  
multiply ideals  $x$  and  $y$       **idealmul**( $nf, x, y, flag$ )  
intersection of ideals  $x$  and  $y$       **idealintersect**( $nf, x, y, flag$ )  
 $n$ -th power of ideal  $x$       **idealpow**( $nf, x, n, flag$ )  
inverse of ideal  $x$       **idealinv**( $nf, x$ )  
divide ideal  $x$  by  $y$       **idealdiv**( $nf, x, y, flag$ )  
Find  $(a, b) \in x \times y$ ,  $a + b = 1$       **idealaddtoone**( $nf, x, \{y\}$ )

### Primes and Multiplicative Structure

factor ideal  $x$  in  $nf$       **idealfactor**( $nf, x$ )  
recover  $x$  from its factorization in  $nf$       **factorback**( $x, nf$ )  
decomposition of prime  $p$  in  $nf$       **idealprimedec**( $nf, p$ )  
valuation of  $x$  at prime ideal  $pr$       **idealval**( $nf, x, pr$ )  
weak approximation theorem in  $nf$       **idealchinese**( $nf, x, y$ )  
give  $bid$  = structure of  $(\mathbf{Z}_K/id)^*$       **idealstar**( $nf, id, flag$ )  
discrete log of  $x$  in  $(\mathbf{Z}_K/bid)^*$       **ideallog**( $nf, x, bid$ )  
**idealstar** of all ideals of norm  $\leq b$       **ideallist**( $nf, b, flag$ )  
add archimedean places      **ideallistarch**( $nf, b, \{ar\}, flag$ )  
init **prmod** structure      **nfmodprinit**( $nf, pr$ )  
kernel of matrix  $M$  in  $(\mathbf{Z}_K/pr)^*$       **nfkermodpr**( $nf, M, prmod$ )  
solve  $Mx = B$  in  $(\mathbf{Z}_K/pr)^*$       **nfsolvemodpr**( $nf, M, B, prmod$ )

### Galois theory over $\mathbf{q}$

initializes a Galois group structure      **galoisinit**( $pol, \{den\}$ )  
action of  $p$  in **nfgaloisconj** form      **galoispermopol**( $G, \{p\}$ )  
identifies as abstract group      **galoisidentify**( $G$ )  
exports a group for GAP or MAGMA      **galoisexport**( $G, flag$ )  
subgroups of the Galois group  $G$       **galoissubgroups**( $G$ )  
subfields from subgroups of  $G$       **galoissubfields**( $G, flag, \{v\}$ )  
fixed field      **galoisfixedfield**( $G, perm, flag, \{v\}$ )  
is  $G$  abelian?      **galoisisabelian**( $G, flag$ )  
abelian number fields      **galoissubcyclo**( $N, H, flag, \{v\}$ )

## Relative Number Fields (rnf)

Extension  $L/K$  is defined by  $g \in K[x]$ . We have  $order \subset L$ .  
absolute equation of  $L$       **rnfequation**( $nf, g, flag$ )  
relative **nfalgtobasis**      **rnfalgtobasis**( $rnf, x$ )  
relative **nfbasistoalg**      **rnfbasistoalg**( $rnf, x$ )  
relative **idealhnf**      **rnfidealhnf**( $rnf, x$ )  
relative **idealmul**      **rnfidealmul**( $rnf, x, y$ )  
relative **idealtwoelt**      **rnfidealtwoelt**( $rnf, x$ )

### Lifts and Push-downs

absolute  $\rightarrow$  relative repres. for  $x$       **rnfeltabstorel**( $rnf, x$ )  
relative  $\rightarrow$  absolute repres. for  $x$       **rnfeltreltoabs**( $rnf, x$ )  
lift  $x$  to the relative field      **rnfeltup**( $rnf, x$ )  
push  $x$  down to the base field      **rnfeltdown**( $rnf, x$ )  
idem for  $x$  ideal: (**rnfideal**)**reltoabs**, **abstorel**, **up**, **down**

### Projective $\mathbf{Z}_K$ -modules, maximal order

relative **polred**      **rnfpolred**( $nf, g$ )  
relative **polredabs**      **rnfpolredabs**( $nf, g$ )  
characteristic poly. of  $a \bmod g$       **rnfcharpoly**( $nf, g, a, \{v\}$ )  
relative Dedekind criterion, prime  $pr$       **rnfdedekind**( $nf, g, pr$ )  
discriminant of relative extension      **rnfdisc**( $nf, g$ )  
pseudo-basis of  $\mathbf{Z}_L$       **rnfpseudobasis**( $nf, g$ )  
relative HNF basis of *order*      **rnfhnfbasis**( $bnf, order$ )  
reduced basis for *order*      **rnflllgram**( $nf, g, order$ )  
determinant of pseudo-matrix  $A$       **rnfdet**( $nf, A$ )  
Steinitz class of *order*      **rnfsteynitz**( $nf, order$ )  
is *order* a free  $\mathbf{Z}_K$ -module?      **rnfisfree**( $bnf, order$ )  
true basis of *order*, if it is free      **rnfbasis**( $bnf, order$ )

### Norms

absolute norm of ideal  $x$       **rnfidealnrmabs**( $rnf, x$ )  
relative norm of ideal  $x$       **rnfidealnrmrel**( $rnf, x$ )  
solutions of  $N_{K/\mathbf{Q}}(y) = x \in \mathbf{Z}$       **bnfisintnorm**( $bnf, x$ )  
is  $x \in \mathbf{Q}$  a norm from  $K$ ?      **bnfisnorm**( $bnf, x, flag$ )  
initialize  $T$  for norm eq. solver      **rnfisnorminit**( $K, pol, flag$ )  
is  $a \in K$  a norm from  $L$ ?      **rnfisnorm**( $T, a, flag$ )

Based on an earlier version by Joseph H. Silverman

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